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The Homogeneous Boltzmann Hierarchy and Statistical Solutions to the Homogeneous Boltzmann Equation

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An existence and uniqueness result for the homogeneous Boltzmann hierarchy is proven, by exploiting the "statistical solutions" to the homogeneous Boltzmann equation.

KEY WORDS: Boltzmann hierarchy; Boltzmann equation; statistical solution.

1. INTRODUCTION

The Boltzmann hierarchy (BH) is a linear system of infinitely many coupled differential equations for the correlation functions of a rarefied gas of particles. It can be derived from the BBGKY hierarchy for n hard spheres of diameter d in the Boltzmann–Grad limit, i.e., letting n go to infinity and d to zero in such a way that the factor nd^2 remains finite. The Boltzmann equation (BE) is the equation satisfied by the one-particle correlation function, under the assumption of "propagation of chaos"; if the initial condition factorizes, the same holds for its time evolution. (For generalities on the BE see refs. 1 and 2.)

We are not concerned here with the rigorous deduction of the BE from the BBGKY hierarchy, which is a deep and difficult problem, till now

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solved only in some special cases, such as in ref. 2 for short times and in ref. 3 for initial conditions which are small perturbations of the vacuum. A preliminary step to the rigorous deduction is the analysis of the limiting problem, that is, the BE or equivalently the BH. One would expect that whenever an existence and uniqueness result is proven for the BE, the same should hold for the BH. Indeed this has already been proven in some cases,[2-4] while others have not yet been studied (e.g., refs. 5 and 6).

Our aim is to prove existence and uniqueness of a class of solutions to the homogeneous Boltzmann hierarchy (HBH), using the fairly complete theory on the homogeneous Boltzmann equation (HBE) developed in refs. 7 and 8. One can obviously prove the existence of factorizing solutions to the BH once results on existence and uniqueness for the BE are available. The uniqueness of this kind of solution is a more delicate point; are there solutions factorizing at time zero and not keeping this character for all later time? In the homogeneous case a negative answer may be given. Indeed, in this paper we prove existence and uniqueness of a class of solutions including the factorizing ones.

Such a result would follow if we could iterate, over any finite time interval, Lanford's argument, which in the space-dependent case is valid only for short times. For that purpose *a priori* estimates are needed on the correlation functions in some norm of exponentially decaying functions as used by Lanford. Because of the complications involved in such an approach, we choose another strategy, exploiting an analogy pointed out in ref. 9 between solutions to the BH and "statistical solutions" to the BE. Statistical solutions of partial differential equations have been investigated in different contexts by various authors (see for example refs. 10 and 11 for the Vlasov equation, and ref. 12 for the fluid dynamics equations).

The plan of this paper is as follows. The definition of statistical solution to the HBE and its relation to the solutions of the HBH is given in Section 2, together with some useful background information. In Section 3 we prove the main theorem of the paper and comment on the approach to equilibrium. Some technical lemmas are left for the final Section 4.

2. ON THE BOLTZMANN HIERARCHY AND STATISTICAL SOLUTIONS TO THE BOLTZMANN EQUATION IN THE SPATIALLY HOMOGENEOUS CASE

For any $j \in \mathbb{N}$, let $V_j \equiv (v_1, ..., v_j)$ represent a j-ple of vectors in $\mathbb{R}^3$ (the velocities of the particles) and let $f_j\colon \mathbb{R}^{3j} \to \mathbb{R}$ be a nonnegative real function with the following properties:

$$f_j(V_j) = f_j(\mathscr{P} V_j) \qquad \text{(symmetry)} \tag{2.1}$$

for all $\mathscr{P}$, where $\mathscr{P}V_j$ is a permutation of the sequence $(v_1,..., v_j)$,

$$\int f_j(V_j)\, dV_j = 1 \qquad \text{(normalization)} \tag{2.2}$$

and

$$f_j(V_j) = \int dv_{j+1} f_{j+1}(V_{j+1}) \qquad \text{(compatibility)} \tag{2.3}$$

Consider the following infinite system of coupled linear differential equations for the f_j:

$$\begin{aligned} \partial_t f_j(V_j, t) &= (C_{j,j+1} f_{j+1})(V_j, t) \\ f_j(V_j, 0) &= f_j(V_j) \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} (C_{j,j+1} f_{j+1})(V_j, t) = \sum_{i=1}^{j} \int_{n \cdot (v_i - v_{j+1}) \geq 0} & dn\, dv_{j+1}\, n \cdot (v_i - v_{j+1}) \\ & \times \{f_{j+1}((V_{j+1})'_i, t) - f_{j+1}(V_{j+1}, t)\} \end{aligned} \tag{2.5}$$

$$(V_{j+1})'_i = (v_1,..., v'_i,..., v_j, v'_{j+1}) \tag{2.6}$$

$$\begin{aligned} v'_i &= v_i - n[(v_i - v_{j+1}) \cdot n] \\ v'_{j+1} &= v_{j+1} + n[(v_i - v_{j+1}) \cdot n] \end{aligned} \tag{2.7}$$

and n is the unit vector in $\mathbb{R}^3$ pointing from the ith to the $(j+1)$th particle.

Equations (2.4) can be interpreted as describing the time evolution of the joint distribution densities f_j associated to a rarefied, homogeneous gas of hard spheres. In other words, $f_j(v_1,..., v_j)$ denotes the probability density of finding any group of j tagged particles with velocities $v_1,..., v_j$. In (2.7), v'_i and v'_{j+1} are the outgoing velocities of two colliding particles with initial velocities v_i and v_{j+1}.

The system of equations (2.4) is known as the *homogeneous Boltzmann hierarchy* (HBH). By (2.1) and (2.7) it is easy to verify that the following quantities are invariant under the evolution (2.4);

$$\int v_i^\alpha f_j(V_j, t)\, dV_j, \qquad \alpha = 0, 1, 2, \quad i = 1,..., j \tag{2.8}$$

Let us now introduce the *homogeneous Boltzmann equation* (HBE)

$$\begin{aligned} \partial_t f(v, t) &= Q(f, f)(v, t) \\ f(v, 0) &= f(v) \\ \int f(v)\, dv &= 1 \end{aligned} \tag{2.9}$$

where f: $\mathbb{R}^3 \to \mathbb{R}^+$, and

$$Q(f, g)(v) = \frac{1}{2} \int_{n \cdot (v - v_*) \geq 0} dn \, dv_* \, n \cdot (v - v_*) \times \{f(v'_*) \, g(v') + f(v') \, g(v'_*) - f(v_*) \, g(v) - f(v) \, g(v_*)\} \quad (2.10)$$

v' and v'_* having the same meaning of outgoing velocities as before. By (2.7) it is easy to see that the following quantities are invariant for Eq. (2.9):

$$\int v^\alpha f(v) \, dv, \qquad \alpha = 0, 1, 2 \quad (2.11)$$

In the case of factorizing distributions, i.e.,

$$f_j(V_j, t) = \prod_{i=1}^{j} f(v_i, t) \quad (2.12)$$

an easy calculation shows that Eqs. (2.4) and (2.9) are equivalent in the following sense:

(a) If there exists a solution to (2.4) of the form (2.12) in $[0, T]$, the f_1 satisfies (2.9) in $[0, T]$.

(b) If there exists a solution $f(v, t)$ to (2.9) in $[0, T]$, the f_j defined in (2.12) satisfies (2.4) as well as the properties (2.1)–(2.3) on that time interval.

In the case of nonfactorizing data, that is, if correlations among particles are present, the study of the Boltzmann hierarchy is important in itself. The aim of this paper is to prove a result of existence and uniqueness of a class of solutions to the HBH (2.4) on R^+. Since there is uniqueness, it is enough to carry out the discussion below on a time interval $[0, T]$ with $T > 0$ arbitrarily fixed.

In the rest of this section, we introduce some notations and definitions and comment on various aspects of the proofs.

Let $\mathcal{N}$ be the set of probability measures on the σ-algebra $\mathcal{B}$ of Borel sets in $\mathbb{R}^3$. Introduce the σ-algebra on $\mathcal{N}$ generated by the sets

$$\{\pi \in \mathcal{N} \mid \pi(E) \leq \lambda\} \quad \text{for } \lambda \text{ real and } E \in \mathcal{B} \quad (2.13)$$

Also let $\mathcal{M}$ be the set of probability measures on $\mathcal{N}$ with this σ-algebra.

Let Σ be the restriction of the σ-algebra of (2.13) to

$$\mathcal{S} = \left\{ f: R^3 \to R^+ \,\middle|\, \int_{R^3} f(v) \, dv = 1 \right\}$$

and let $\mathcal{M}^1(\Sigma)$ be the set of probability measures on Σ. Define

$$\|f\|_\kappa = \int (1+|v|^2)^{\kappa/2} |f(v)| \, dv \tag{2.14}$$

and

$$\mathcal{M}^1_\kappa(\Sigma) = \{\mu \in \mathcal{M}^1(\Sigma) \mid \mu(H(f)) < \infty; \mu((\|f\|_\kappa)^j) < C^j_\mu, j = 1, 2,...\} \tag{2.15}$$

Here

$$H(f) = \int f \ln f \, dv$$

is the entropy, and C stands for a positive constant. The dependence of C on parameters will be indicated when it is relevant.

Given any $\mu \in \mathcal{M}^1_\kappa(\Sigma)$, define the family $\mathcal{F} = (f_j)_{\mathbb{N}}$ by

$$f_j(V_j) = \int \mu(df) \prod_1^j f(v_i) \tag{2.16}$$

The f_j satisfy properties (2.1)–(2.3) together with

$$\frac{1}{j} \int dV_j f_j(V_j) \ln f_j(V_j) < C_{\mathcal{F}}, \qquad j = 1, 2,... \tag{2.17}$$

$$\int dV_j f_j(V_j) \prod_{i=1}^j (1+|v_i|^2)^{\kappa/2} < C^j_{\mathcal{F}}, \qquad j = 1, 2,... \tag{2.18}$$

as follows by Fubini's theorem and Jensen's inequality applied to the convex function $x \ln x$.

We are also interested in getting a one-to-one correspondence between a suitable set of sequences $\mathcal{F} = (f_j)_{\mathbb{N}}$ and the probability measures μ in $\mathcal{M}^1_\kappa(\Sigma)$. By the Hewitt–Savage theorem[13] given a family $\mathcal{F} = (f_j)_{\mathbb{N}}$ satisfying properties (2.1)–(2.3), then there exists a unique measure μ belonging to $\mathcal{M}$ such that

$$f_j(V_j) \, dV_j = \int \mu(d\nu) \prod_{i=1}^j d\nu(v_i) \tag{2.19}$$

If we make the assumption (2.17), then μ is supported on absolutely continuous probability measures. That is so because under (2.17)

$$H(\mu) := \lim \frac{1}{j} \int dV_j f_j(V_j) \ln f_j(V_j)$$

exists, and

$$H(\mu) = \int d\mu(f)\, H(f) \leqslant C_{\mathscr{F}} \tag{2.20}$$

For a proof of (2.20) see ref. 15, Proposition 5 and ref. 14, Lemma 10. This allows us to write (2.19) in terms of densities. Furthermore, under (2.18) μ belongs to $\mathscr{M}^1_\kappa(\Sigma)$, since the first condition of (2.15) holds by (2.20), while the second one follows from (2.18).

Thus, any family $\mathscr{F} = (f_j)_{\mathbb{N}}$ satisfying (2.1)–(2.3), (2.17), and (2.18) can via the Hewitt–Savage theorem be expressed in a unique way by (2.16) with a measure μ belonging to $\mathscr{M}^1_\kappa(\Sigma)$.

Let us now introduce the space

$$\mathscr{S}_\kappa = \{ f \in \mathscr{S};\, H(f) < \infty,\, \| f \|_\kappa < \infty \} \tag{2.21}$$

The HBE (2.9) has a unique solution f with $f_t \in \mathscr{S}_\kappa$ for $t > 0$, when the initial value f_0 is in $\mathscr{S}_\kappa$ and $\kappa \geqslant 4$ (see ref. 8). Moreover, using the collisional estimate

$$\begin{aligned} |v'|^s + |v'_*|^s - |v|^s - |v_*|^s \leqslant{}& K_s(|v|^{s-1}\, |v_*| + |v|\, |v_*|^{s-1} \cos\theta \sin\theta \\ &- C_s(|v|^s + |v_*|^s) \cos^2\theta \sin^2\theta, \qquad s > 2 \end{aligned}$$

from ref. 16, Theorem 2, the following bound for such solutions can be proved:

$$\| f_t \|_\kappa \leqslant C(T, \kappa)\, \| f \|_\kappa^{\kappa - 1} \tag{2.22}$$

with $C(T, \kappa)$ only depending on T and κ.

Let β be a probability measure in $\mathscr{M}^1_\kappa(\Sigma)$ for $\kappa \geqslant 4$. Define by means of the Boltzmann flow $\mathscr{T}_t$ a corresponding time-evolved measure as follows:

$$\beta_t(A) = \beta(\mathscr{T}_{-t} A) \tag{2.23}$$

for $t \in [0, T]$, $A \in \Sigma$, and

$$\mathscr{T}_{-t} A = \{ f \in \mathscr{S}_\kappa \mid f_t := \mathscr{T}_t f \in A \} \tag{2.24}$$

Setting

$$f_{j,t}(V_j) := \int \beta_t(df) \prod_{i=1}^{j} f(v_i) \tag{2.25}$$

and

$$f_j(\cdot, t) := f_{j,t}(\cdot)$$

the $f_{j,t}$ satisfy the HBH (2.4). The $f_{j,t}$ also enjoy the properties (2.17) and (2.18) over the interval $[0, T]$ since $\beta \in \mathscr{M}^1_\kappa(\Sigma)$. Indeed,

$$\beta_t(H(f)) = \beta(H(f_t)) \leqslant \beta(H(f)) < \infty$$

and by (2.22)

$$\beta_t(\|f\|_\kappa)^j) = \beta((\|f_t\|_\kappa)^j) \leqslant (C(T, \kappa)\, C_\beta^{\kappa-1})^j \tag{2.26}$$

So there exists—by a natural construction—a solution to (2.4) which satisfies (2.17) and (2.18). The uniqueness of that kind of solution is the main problem of this paper.

As mentioned in the introduction, a direct investigation seems complicated. Instead we choose an approach using the previously introduced probability measures related to the solutions of (2.4). With that in mind we first discuss the evolution equation for β_t of (2.23) on an appropriate algebra of test functions. Then we show that this new evolution problem is closely connected to our initial HBH problem (2.4).

In fact they are equivalent in a way that will be specified at the end of this section.

The space of test functions is the following: Let G_j: $V_j \in \mathbb{R}^{3j} \to G_j(V_j) \in \mathbb{R}$ be an $\mathscr{L}^\infty$-function. Define the class of functions on $\mathscr{S}_\kappa$ (for any $j \in \mathbb{N}$)

$$\begin{aligned} F &= \bigcup_{j \in \mathbb{N}} F_j \\ F_j &= \left\{\phi : \mathscr{S}_\kappa \to \mathbb{R} \mid \phi(f) = \int dV_j\, G_j(V_j) \prod_{i=1}^{j} f(v_i)\right\} \end{aligned} \tag{2.27}$$

By the BE (2.9),

$$\begin{aligned} \frac{d}{dt} \prod_{i=1}^{j} f_t(v_i) = {} & \sum_{i=1}^{j} \int dn\, dv_{j+1}\, n \cdot (v_i - v_{j+1}) \\ & \times \prod_{k \neq i} f_t(v_k) \{ f_t(v_i') \cdot f_t(v_{j+1}') - f_t(v_i) \cdot f_t(v_{j+1}) \} \end{aligned}$$

This together with the change of variables

$$(v_i', v_{j+1}') \to (v_i, v_{j+1})$$

gives that

$$\frac{d}{dt} \beta_t(\phi) = \beta_t(L\phi) \tag{2.28}$$

where

$$\beta_t(\phi) = \int \beta_t(df)\,\phi(f)$$
$$L\phi(f) = \int dV_{j+1}\, G^*_{j+1}(V_{j+1}) \prod_{i=1}^{j+1} f(v_i) \tag{2.29}$$

and

$$G^*_{j+1}(V_{j+1}) = \sum_{i=1}^{j} \int dn\, n\cdot(v_i - v_{j+1})\cdot[G_j(v_1,..., v_i',..., v_j) - G_j(V_j)] \tag{2.30}$$

Notice that $L\phi \in \mathscr{L}^1(\mu)$ for any $\mu \in \mathscr{M}^1_\kappa(\Sigma)$, since $G_j \in \mathscr{L}^\infty(\mathbb{R}^{3j}, \mathbb{R})$, and the f's are in $\mathscr{S}_\kappa$ with $\kappa \geqslant 4$. In particular, $L\phi \in \mathscr{L}^1(\beta_t)$. Moreover, it follows from definition (2.29) that

$$L\phi(f) = \lim_{h\to 0} \frac{\phi(f_h) - \phi(f)}{h} \tag{2.31}$$

The algebra F of test functions is large enough to determine the measure μ in a unique way. Indeed, let K_α be the closure with respect to the weak topology of measures of the set of $f \in \mathscr{S}_\kappa$ with $\|f\|_\kappa < \alpha$. Then K_α is compact. Consider the subalgebra F_α of $\mathscr{C}(K_\alpha, \mathbb{R})$ (the set of continuous functions on K_α) consisting of ϕ's in F_j defined by bounded continuous functions G_j. The identity belongs to F_α; moreover, for any f, g in K_α such that $f \neq g$, there is $\phi \in F_\alpha$ such that $\phi(f) \neq \phi(g)$. Thus, by Stone's theorem, F_α is dense in the uniform topology of $\mathscr{C}(K_\alpha, \mathbb{R})$. Thus F_α uniquely determines the measure μ_α, restriction of μ to K_α. Moreover, by the condition $\mu(\|f\|_\kappa) < +\infty$ it follows that for any set $A \subset \Sigma$

$$\mu(A) = \mu(A \cap K_\alpha) + \mu(A \backslash K_\alpha) \leqslant \mu_\alpha(A) + C_0/\alpha$$

that is, μ is known once μ_α is determined.

We are now ready to describe the evolution equation for β_t and related measures. Given a measure $\mu_0 \in \mathscr{M}^1_\kappa(\Sigma)$, consider the following evolution problem:

$$\frac{d}{dt}\mu_t(\phi) = \mu_t(L\phi), \qquad \phi \in F$$
$$\mu\,|_{t=0} = \mu_0 \tag{2.32}$$

Any differentiable function $t \to \mu_t$ from the time interval $[0, T]$ to $\mathscr{M}^1_\kappa(\Sigma)$ and satisfying (2.32) is called a *statistical solution* to the HBE (2.9). By the previous discussion, β_t of (2.23) is a statistical solution with initial value $\beta \in \mathcal{M}_\kappa^1(\Sigma)$.

Consider also the HBH (2.4). Multiply (2.4) by a function $G_j \in \mathcal{S}^\infty(\mathbb{R}^{3j}, \mathbb{R})$ and integrate both sides with respect to dV_j. This gives a weak form of the HBH, which after a change of variables can be written as follows:

$$\begin{aligned} \frac{d}{dt} \langle f_{j,t}, G_j \rangle &= \langle f_{j+1,t}, G_{j+1}^* \rangle \\ \langle f_{j,0}, G_j \rangle &= \langle f_j, G_j \rangle \end{aligned} \tag{2.33}$$

Here

$$\langle a_j, b_j \rangle = \int dV_j \, a(V_j) \, b(V_j)$$

and G_{j+1}^* is defined in (2.29).

Using (2.19), define μ_0 from the initial values $(f_{j,0})_{\mathbb{N}}$ in (2.33). The solution β_t of (2.32) with initial value μ_0 defines via (2.19) a solution of (2.33). More generally the following holds.

Set

$$\tilde{\mathcal{S}}_\kappa = \{(f_j)_{j \in \mathbb{N}}; \text{properties (2.1)–(2.3), (2.17), and (2.18) hold}\}$$

and choose $\kappa \geq 4$. A family $(f_j)_N$: $[0, T] \to \tilde{\mathcal{S}}_\kappa$ is a solution to (2.33) *if and only if* μ_t is a solution to (2.32) belonging to $\mathcal{M}_\kappa^1(\Sigma)$ over $[0, T]$, where

$$f_{j,t}(V_j) = \int \mu_t(df) \prod_{i=1}^{j} f(v_i) \tag{2.34}$$

3. STATEMENT AND PROOF OF THE MAIN THEOREM

The main result of this paper is the following.

Theorem 3.1. There exists a unique solution

$$(f_j)_{\mathbb{N}}: \mathbb{R}^+ \to \tilde{\mathcal{S}}_\kappa$$

to the weak homogeneous Boltzmann hierarchy (2.33) when $\kappa \geq 4$. This is given by

$$f_{j,t}(V_j) = \int \beta_t(df) \prod_{i=1}^{j} f(v_i), \qquad j = 1, 2, \ldots$$

with β_t defined by (2.23).

We already know that the family $(f_j)_{\mathbb{N}}$ of (2.25) defines a strong solution to (2.33). So, by the discussion at the end of Section 2, Theorem 3.1 is a consequence of the following proposition.

Proposition 3.1. There exists a unique solution to the initial value problem (2.32) which belongs to $\mathcal{M}^1_\kappa(\Sigma)$ for positive time. This solution is for $t \in \mathbb{R}^+$ given by β_t of (2.23).

Thus it only remains to prove Proposition 3.1. For this we introduce a modified Boltzmann equation as follows. Choose an arbitrary positive T and divide the time interval $[0, T]$ into N intervals of width $\varepsilon = T/N$ ($N \in \mathbb{N}$). For any $t \in [0, T]$, let us consider the evolution equation:

$$\begin{aligned} &\frac{d}{dt} f_t^N(v) = Q^N(f^N_{[t]}, f^N_{[t]})(v) \\ &f_0^N(v) = f(v) \in \mathcal{S} \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} Q^N(f, g)(v) = \frac{1}{2} \int_{n \cdot (v - v_1) \geqslant 0} & dn\, dv_1\, n \cdot (v - v_1)\, \kappa^N(v, v_1) \\ & \times \{f(v_1')\, g(v') + f(v')\, g(v_1') - f(v)\, g(v_1) - f(v_1)\, g(v)\} \end{aligned} \tag{3.2}$$

$$\kappa^N(v, v_1) = \begin{cases} 1 & \text{if } \ |v - v_1| \leqslant \ln \ln N \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

$$[t] = \max_{k \in \mathbb{N}} \{k\varepsilon : k\varepsilon < t\} \tag{3.4}$$

The above dynamics has been introduced for the following reason. For $\phi \in F$ set

$$\begin{aligned} U_t^N \phi(f) &= \phi(f_t^N) \\ U_t \phi(f) &= \phi(f_t) \end{aligned} \tag{3.5}$$

Then, while $U_t \phi$ does not (in general) belong to F, $U_t^N \phi$ does because of its particular dependence upon the initial data, which at any time has a product structure in terms of f, i.e., of f^N at time zero. This will be of central importance in the proof of Proposition 3.1. Furthermore, Eq. (3.1) enjoys the following properties.

(i) It has the same invariants as Eq. (2.9).

(ii) It follows by the definition of Q^N that

$$\|Q^N(f, g)\|_0 \leqslant C \ln \ln N \, \|f\|_0 \, \|g\|_0 \tag{3.6}$$

(iii) As a consequence of the nonnegativity of f_t^N

$$\|f_t^N\|_0 = \|f\|_0 \quad \text{in} \quad [0, T] \tag{3.7}$$

Indeed, if $f \geqslant 0$, then at time $t = \varepsilon$,

$$f_\varepsilon^N = f + \varepsilon Q^N(f, f) = f(1 - \varepsilon L^N(f)) + \varepsilon J^N(f, f)$$

Here J^N is the "gain" and fL^N the "loss" part of the collision operator, i.e., the first and the second couple of terms, respectively, in (3.2). By (3.6)

$$f_\varepsilon^N \geqslant f(1 - \varepsilon C \ln \ln N \|f\|_0) \geqslant 0 \tag{3.8}$$

if ε is sufficiently small. Since $\int f\,dv$ is an invariant for (3.1), we can iterate (3.8) up to time T.

(iv) In exactly the same way as for Eq. (2.9), the bound (2.22) can be proven for f_t^N on $[0, T]$.

The proof of Proposition 3.1 depends on the following two technical lemmas, which will be proved in Section 4.

Lemma 3.1. If $f \in \mathscr{S}_\kappa$, $\kappa \geqslant 4$, then for $t \in [0, T]$,

$$\|f_t^N - f_t\|_0 \leqslant C \exp(C_1 \|f\|_4^3)/\ln \ln N$$

Let $(f_h)_t^N$ denote the solution to (3.1) with initial value f_h, where f_h is the solution to (2.9) at time $h > 0$ with initial value f. Set

$$\mathscr{L} f_s^N = \lim_{h \to 0} \frac{(f_h)_s^N - f_s^N}{h} \tag{3.9}$$

$$\mathscr{D} f_s^N = \frac{f_{s+\varepsilon}^N - f_s^N}{\varepsilon} \tag{3.10}$$

We note that $\mathscr{L}$ plays the role of a derivative with respect to the initial conditions, while $\mathscr{D}$ is the usual discrete time derivative.

Lemma 3.2. If $f \in \mathscr{S}_\kappa$, $\kappa \geqslant 4$, then for $t \in [0, T]$, $\|(\mathscr{L} - \mathscr{D}) f_{[t]}^N\|_2 \leqslant \exp(C \|f\|_3^2)\, \varphi(N)\, \|f\|_4$, for some function φ of N (depending on T), such that

$$\lim_{N \to \infty} \varphi(N) = 0$$

Proof of Proposition 3.1. Suppose that there is a solution μ_t to

(2.32) different from β_t as defined in (2.23) and belonging to $\mathscr{M}^1_\kappa(\Sigma)$ on $[0, T]$. Then

$$(\mu_t - \beta_t)(\phi) = \mu_t(\phi) - \mu_0(U_t^N\phi) + \mu_0(U_t^N\phi - U_t\phi) \tag{3.11}$$

The last term in (3.11) can be controlled using (3.7) and Lemma 3.1,

$$\begin{aligned} &|U_t^N\phi(f) - U_t\phi(f)| \\ &\leqslant \sum_{k=1}^{j} \int dV_j \left| G_j(V_j) \prod_{i=1}^{k-1} f_t^N(v_i) \prod_{i=k+1}^{j} f_t(v_i)[f_t^N(v_k) - f_t(v_k)] \right| \\ &\leqslant j\, \|G_j\|_\infty\, \|f_t^N - f_t\|_0 \leqslant C \exp(C\, \|f\|_4^3)/\ln \ln N \end{aligned} \tag{3.12}$$

Let us now evaluate the rest of the right-hand side in (3.11), for the moment only considering rational times $t = n\varepsilon$ for some $n \in \mathbb{N}$. Then

$$\begin{aligned} &\mu_t(\phi) - \mu_0(U_t^N\phi) \\ &= \sum_{k=0}^{n-1} \mu_{(n-k)\varepsilon} U_{k\varepsilon}^N \phi - \mu_{(n-k-1)\varepsilon} U_{(k+1)\varepsilon}^N \phi \\ &= \int_0^t ds \left\{ \frac{-d}{ds} \mu_{t-s}(U_{[s]}^N\phi) - \mu_{t-[s]-\varepsilon} \left[\frac{(U_{[s]+\varepsilon}^N - U_{[s]}^N)}{\varepsilon} \phi \right] \right\} \end{aligned} \tag{3.13}$$

by adding and subtracting in the sum the quantity $\mu_{(n-k-1)\varepsilon}(U_{k\varepsilon}^N\phi)$. As we remarked after (3.5), $U_s^N\phi \in F$ for any $s \leqslant T$, and so

$$\begin{aligned} &\mu_t(\phi) - \mu_0(U_t^N\phi) \\ &= \int_0^t ds \{\mu_{t-s}(LU_{[s]}^N\phi) - \mu_{t-[s]-\varepsilon}(DU_{[s]}^N\phi)\} \\ &= \int_0^t ds\, \mu_{t-s}[(L-D)\, U_{[s]}^N\phi] + \int_0^t ds \int_s^{[s]+\varepsilon} d\tau\, \mu_{t-\tau}[L(DU_{[\tau]}^N)] \\ &:= \mathscr{T}_1 + \mathscr{T}_2 \end{aligned} \tag{3.14}$$

Here, by (2.31)

$$LU_{[s]}^N\phi(f) = \lim_{h\to 0} \frac{U_{[s]}^N\phi(f_h) - U_{[s]}^N\phi(f)}{h} \tag{3.15}$$

and

$$DU_{[s]}^N\phi(f) = \frac{1}{\varepsilon}\, [U_{[s]+\varepsilon}^N\phi(f) - U_{[s]}^N\phi(f)] \tag{3.16}$$

Evidently (3.14) holds strictly, once it is proved that all terms make sense. This will be done next starting with $\mathscr{T}_1$. We have

$$LU^N_{[s]}\phi(f)=\int dV_j\left\{G_j(V_j)\lim_{h\to 0}\frac{1}{h}\left[\prod_{i=1}^{j}(f_h)^N_{[s]}(v_i)-\prod_{i=1}^{j}f^N_{[s]}(v_i)\right]\right\}$$

$$=\sum_{i=1}^{j}\int dV_j\left\{G_j(V_j)\prod_{\substack{k=1\\k\neq i}}^{j}f^N_{[s]}(v_k)\,\mathscr{L}f^N_{[s]}(v_i)\right\} \tag{3.17}$$

and analogously

$$DU^N_{[s]}\phi(f)=\sum_{i=1}^{j}\int dV_j\left\{G_j(V_j)\prod_{k=1}^{i-1}f^N_{[s]+\varepsilon}(v_k)\prod_{k=i+1}^{j}f^N_{[s]}(v_k)\,\mathscr{D}f^N_{[s]}(v_i)\right\}$$

$$=\sum_{i=1}^{j}\int dV_j\,G_j(V_j)\prod_{\substack{k=1\\k\neq i}}^{j}f^N_{[s]}(v_k)\,\mathscr{D}f^N_{[s]}(v_i)$$

$$+\sum_{i=1}^{j}\int dV_j\,G_j(V_j)\left\{\varepsilon\sum_{\nu=1}^{i-1}\prod_{k=1}^{\nu-1}f^N_{[s]}(v_k)\right.$$

$$\left.\times\,Q^N(f^N_{[s]})(v_\nu)\prod_{k=\nu+1}^{i-1}f^N_{[s]+\varepsilon}(v_k)\right\}$$

$$\times\prod_{k=i+1}^{j}f^N_{[s]}(v_k)\,\mathscr{D}f^N_{[s]}(v_i) \tag{3.18}$$

Hence by (3.6) and (3.7) it follows that

$$|(L-D)U^N_{[s]}\phi|\leqslant j\,\|G_j\|_\infty\,[\|(\mathscr{L}-\mathscr{D})f^N_{[s]}\|_0+\varepsilon j(\ln\ln N)\,\|\mathscr{D}f^N_{[s]}\|_0] \tag{3.19}$$

Since $\mathscr{D}f^N_{[s]}=Q^N(f^N_{[s]})$, again by (3.6), the second term in (3.19) tends to zero with ε uniformly in f and $[s]\leqslant T$. Taking into account Lemma 3.2 and (3.19) and recalling that $\varepsilon=T/N$, we thus have

$$\mathscr{T}_1\leqslant\psi(T,N,j)\left[\int_0^t ds\,\mu_{t-s}\{\|f\|_4\exp(C\,\|f\|_3^2)\}+1\right] \tag{3.20}$$

where ψ is a function such that

$$\lim_{N\to\infty}\psi(T,N,j)=0$$

Since μ_{t-s} belongs to $\mathscr{M}^1_\kappa(\Sigma)$, the integral in (3.20) can be bounded by a constant uniformly in $[0,T]$ by writing the exponential as its Taylor expansion. Then $\mathscr{T}_1$ converges to zero as N tends to infinity.

As for $\mathscr{T}_2$, it follows from (3.17), (3.18), and Lemma 3.2 that

$$|L(DU^N_{[s]}\phi)(f)| \leqslant Cj^2 \|g_j\|_\infty (\ln\ln N)\{\exp(C_1 \|f\|_3^2)\,\varphi(N)\,\|f\|_4 + \ln\ln N\}$$

Since $\mu_{t-\tau} \in \mathscr{M}^1_\kappa(\Sigma)$ and $[s]+\varepsilon-s \leqslant T/N$, $\mathscr{T}_2$ converges to zero as N tends to infinity. The convergence of $\mathscr{T}_1$ and $\mathscr{T}_2$ to zero when N tends to infinity together with (3.12) allows us to conclude that μ_t coincides with β_t for any rational time $t = n\varepsilon$. By a density argument this result can be extended to any real time $t \in [0, T]$. ∎

Remark on the Asymptotic Behavior. Our solutions to the HBH are of the kind

$$f_{j,t}(V_j) = \int \beta_t(df) \prod_{i=1}^{j} f(v_i)$$

β_t being defined in (2.23). Under the assumption of finite entropy $H(f)$, it is known that any solution to the HBE converges weakly, as time goes to infinity, to a Maxwell distribution

$$M(v; u, T) = (2\pi T)^{-3/2} \exp[-(v-u)^2/2T]$$

where T and u, temperature and mean velocity, depend on f (the density is fixed by the normalization condition).

Therefore, since $\beta_t \in \mathscr{M}^1_\kappa(\Sigma)$, a positive measure μ_∞ on $\mathbb{R}^+ \times \mathbb{R}^3$ exists, such that

$$\lim_{t\to\infty} f_{j,t} = \int \mu_\infty(du, dT) \prod_{i=1}^{j} M(u, T)$$

Here the limit is to be interpreted in the weak sense. The measure μ_∞ can be determined from the initial measure μ,

$$\int \mu_\infty(dT, du)\,\varphi(u, T) = \int \mu(df)\,\varphi(u(f), T(f))$$

(for more details, see ref. 4).

4. PROOF OF THE AUXILIARY LEMMAS

Proof of Lemma 3.1. Let $\bar{f}_t$ be the solution to the following Cauchy problem:

$$\frac{d}{dt}\bar{f}_t(v) = Q^N(\bar{f}_t, \bar{f}_t)(v)$$
$$\bar{f}_0(v) = f(v) \in \mathscr{S}_\kappa \tag{4.1}$$

with Q^N defined in (3.2).

By ref. 8, we know that there is a unique solution to (4.1), which belongs to $\mathscr{S}_\kappa$ for any $t \in [0, T]$. Moreover, using (2.22), it follows that

$$\|\bar{f}_t - f_t\|_0 \leqslant C \exp(C_1 \|f\|_4^3)/\ln \ln N \tag{4.2}$$

Here f_t is the solution to (2.9) with the same initial value f. Since

$$\|f_t^N - f_t\|_0 \leqslant \|f_t^N - \bar{f}_t\|_0 + \|\bar{f}_t - f_t\|_0 \tag{4.3}$$

by (4.2) we are left with the analysis of $\|f_t^N - \bar{f}_t\|_0$. By (3.6) and (3.7)

$$\begin{aligned} \|f_t^N - \bar{f}_t\|_0 &\leqslant \int_0^t ds\, \|Q^N(f_{[s]}^N + \bar{f}_s, f_{[s]}^N - \bar{f}_s)\|_0 \\ &\leqslant 2c \ln \ln N \int_0^t ds\, \{\|f_s^N - \bar{f}_s\|_0 + \|f_s^N - f_{[s]}^N\|_0\} \\ &\leqslant 2c \ln \ln N \int_0^t ds \left\{\|f_s^N - \bar{f}_s\|_0 + \frac{\|Q^N(f_{[s]}^N, f_{[s]}^N)\|_0}{N}\right\} \\ &\leqslant 2c \ln \ln N \left\{\int_0^t ds\, \|f_s^N - \bar{f}_s\|_0 + \frac{Tc \ln \ln N}{N}\right\} \end{aligned} \tag{4.4}$$

since $s - [s] < T/N$. But (4.4) implies that

$$\|f_t^N - \bar{f}_t\|_0 \leqslant \frac{C}{\ln \ln N}, \qquad t \leqslant T$$

thus completing the proof of the lemma. ▮

Proof of Lemma 3.2. Let us start from

$$\begin{aligned} \mathscr{L} f_{[s]}^N &= \mathscr{L} f + 2 \int_0^{[s]} d\tau\, Q^N(f_{[\tau]}^N, \mathscr{L} f_{[\tau]}^N) \\ \mathscr{D} f_{[s]}^N &= \mathscr{D} f + \int_0^{[s]} d\tau\, Q^N(f_{[\tau]+\varepsilon}^N + f_{[\tau]}^N, \mathscr{D} f_{[\tau]}^N) \end{aligned} \tag{4.5}$$

Here we have used that

$$f_{[s]+\varepsilon}^N = (f_\varepsilon^N)_{[s]}^N$$

With $\delta = \mathscr{L} - \mathscr{D}$, it follows that

$$\delta f_{[s]}^N = \delta f + 2 \int_0^{[s]} d\tau\, Q^N(f_{[\tau]}^N, \delta f_{[\tau]}^N) + \int_0^{[s]} d\tau\, Q^N(f_{[\tau]+\varepsilon}^N - f_{[\tau]}^N, \mathscr{D} f_{[\tau]}^N) \tag{4.6}$$

To obtain an N-independent bound for $\|\delta f^N_{[s]}\|_2$ over $[0, T]$, we need to pass to absolute values in (4.6),

$$\|\delta f^N_{[s]}\|_2 \leqslant \|\delta f\|_2 + 2\int_0^{[s]} d\tau \int dv \operatorname{sign} \delta f^N_{[\tau]}(v) \times (1+|v|^2)\, Q^N(f^N_{[\tau]}, \delta f^N_{[\tau]})(v) + \int_0^{[s]} d\tau\, \|Q^N(f^N_{[\tau]} - f^N_{[\tau]+\varepsilon}, \mathscr{D} f^N_{[\tau]}\|_2 \tag{4.7}$$

Let us first consider the second term in the right-hand side of (4.7). We use the technique of ref. 17, and give the important steps for the sake of completeness. Splitting Q^N into its four terms, two each from the gain and the loss terms, and recalling that f^N_t is positive on $[0, T]$, we have

$$\begin{aligned} 2\int dv\, (1+|v|^2) \operatorname{sign} \delta f^N_{[\tau]} Q^N(f^N_{[\tau]}, \delta f^N_{[\tau]}) &\leqslant 2\int dv\, (1+|v|^2)\, Q^N(f^N_{[\tau]}, |\delta f^N_{[\tau]}|)(v) \\ &\quad + 2c \int dv\, dv_1\, (1+|v|^2)\, |v-v_1|\, \kappa^N(v, v_1)\, f^N_{[\tau]}(v)\, |\delta f^N_{[\tau]}(v_1)| \\ &\leqslant 0 + C\, \|f^N_{[\tau]}\|_3\, \|\delta f^N_{[\tau]}\|_2 \end{aligned} \tag{4.8}$$

which by (2.22) implies

$$\text{“(4.8)”} \leqslant C\, \|f\|_3^2\, \|\delta f^N_{[\tau]}\|_2 \tag{4.9}$$

Next an estimate of the first term in (4.7) gives

$$\begin{aligned} \|\delta f\|_2 &= \|Q(f, f) - Q^N(f, f)\|_2 \\ &\leqslant \frac{C}{\ln \ln N} \int dv\, dv_1\, dn\, (1+|v|^2)\, |v-v_1|^2\, |f(v')\, f(v_1') - f(v)\, f(v_1)| \\ &\leqslant \frac{C}{\ln \ln N} \int dv\, dv_1\, [1+|v|^2+|v_1|^2]\, |v-v_1|^2\, f(v)\, f(v_1) \\ &\leqslant \frac{C}{\ln \ln N}\, \|f\|_4\, \|f\|_2 \end{aligned} \tag{4.10}$$

Let us finally evaluate the last term in (4.7),

$$\begin{aligned} &\|Q^N(f^N_{[\tau]} - f^N_{[\tau]+\varepsilon}, \mathscr{D} f^N_{[\tau]})\|_2 \\ &\quad = \varepsilon\, \|Q^N(Q^N(f^N_{[\tau]}, f^N_{[\tau]}), Q^N(f^N_{[\tau]}, f^N_{[\tau]}))\|_2 \\ &\quad \leqslant CT\, \|f\|_2\, (\ln \ln N)^3/N \end{aligned} \tag{4.11}$$

Collecting the estimates (4.9)–(4.11), we get

$$\|\delta f^N_{[s]}\|_2 \leqslant \exp(C\,\|f\|_3^2)\,C\,\|f\|_4/\ln\ln N \quad \blacksquare$$

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